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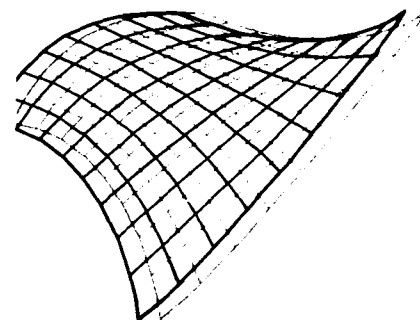
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HIERARCHICAL FINITE ELEMENT APPROACHES,  
ERROR ESTIMATES AND ADAPTIVE REFINEMENT,

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## HIERARCHICAL FINITE ELEMENT APPROACHES, ERROR ESTIMATES AND ADAPTIVE REFINEMENT

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### 1. INTRODUCTION

Despite a continuing effort to identify optimal finite element grids most of the finite element computations today still rely on an a-priori mesh design based on the user's intuition and experience. Once the mesh is designed, however, there seems to be growing evidence that high order isoparametric elements provide a better refinement process than mesh subdivision.

— This paper is concerned with the identification of the discretization error in finite element solution and the definition of optimal refinement processes. The advantages and limitations of the hierarchical approach [58, 36-40] will be discussed and it will be shown how the intelligent enrichment of the finite element grid can be left to the computer if a capacity for a-posteriori error estimation exists within the finite element code [1-14, 37-40, 46-49].

### 2. HIERARCHICAL FINITE ELEMENTS

#### 2.1 *Hierarchical Shape Functions*

The concept of hierarchical finite elements dates from 1970 and these were first introduced [58] with the objective of creating elements that would allow an easy transition from a region where a finite element solution required a high degree of refinement to a region where a lower degree of refinement was sufficient. Other advantages soon become apparent and it will be shown here that the hierarchical concept is very powerful in allowing an error indication capability that can be used for adaptive mesh refinement [36-39, 46-49].

We will begin by defining hierarchical finite elements as those in which successive refinements are additive in the manner of additional terms in a Fourier series. It follows that the "stiffness" matrix corresponding to the hierarchical element at a certain level of refinement is a sub-matrix of the "stiffness"

matrix corresponding to a higher level of refinement.

This leads to matrix approximation equations of the type:

$$K_{11} \underline{a}_1^{(1)} + \underline{q}_1 = \underline{0} \quad (2.1)$$

and

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \underline{a}_1^{(2)} \\ \underline{a}_2^{(2)} \end{Bmatrix} + \begin{Bmatrix} \underline{q}_1 \\ \underline{q}_2 \end{Bmatrix} = \underline{0} \quad (2.2)$$

where (2.1) is the finite element equilibrium equation corresponding to a certain formulation, and equation (2.2) is the same equation corresponding to a higher order or refinement. The matrices  $K_{11}$  and  $\underline{q}_1$  remain unchanged.

Consider solving the linear differential equation

$$A(\phi) \equiv L \phi + q = 0 \quad \text{in } \Omega \quad (2.3)$$

with boundary condition

$$B(\phi) \equiv \bar{L} \phi + s = 0 \quad \text{on } \Gamma \quad (2.4)$$

where  $\Gamma$  is boundary of  $\Omega$ .

Approximate the solution  $\phi$  of (2.3) (2.4) by the solution  $\hat{\phi}$  in the form

$$\hat{\phi} = \sum_{m=1}^M a_m N_m \quad (2.5)$$

with a proper choice of the basis function  $N_i$ ,  $i=1,2,\dots,m$ .

The coefficient  $a_m$  will be determined from the condition that

$$\int_{\Omega} W_e [L\hat{\phi} + q] d\Omega + \int_{\Gamma} \bar{W}_e [\bar{L}\hat{\phi} + s] d\Gamma = 0 \quad (2.6)$$

holds for all  $W_e$  and  $\bar{W}_e$ , where  $W_e$  and  $\bar{W}_e$  are functions suitable for the problem (2.3) (2.4). In addition (2.6) is meant in a generalized way so that it can be used even if  $LN_i$  does not exist in a classical sense (see [59]).

The p-version of the finite element method is defined for a sequence of solutions as the enrichment of the trial and test function set through the introduction of shape functions  $N_m$  corresponding to higher order polynomial degree,  $p$ , while the  $h$  version is defined as the approach equivalent to the reduction of the finite element mesh size  $h$ , maintaining constant the polynomial order [8-9, 46-49].

In 1-D the p-version of the finite element method involves

the addition to the linear element of quadratic, cubic, etc. trial functions as shown in Figure 1. Considering the trial functions expressed as a function of the local element coordinates with  $-1 \leq \xi \leq 1$  we may have for the p-version the following hierarchical shape functions. For the linear terms

$$\begin{aligned} N_0 &= \frac{1-\xi}{2} \\ N_1 &= \frac{1+\xi}{2} \end{aligned} \quad (2.7)$$

For the quadratic the 'obvious' form will be a quadratic that goes to zero at points  $\xi = -1$  and  $\xi = 1$  so that it does not interfere with the  $a_0$  and  $a_1$  coefficients. Thus

$$N_2 = (\xi - 1)(\xi + 1) \quad (2.8)$$

For the cubic and higher order elements the only restriction is that the shape functions will go to zero at  $\xi = -1$ ,  $\xi = 1$ , so that we write for the p-th order

$$N_p = (\xi - 1)(\xi + 1)\xi^{p-2}, \quad p \geq 2 \quad (2.9)$$

The above set is obviously not unique and many alternatives are possible.

A convenient form of hierarchical functions is given in Peano et al. [38] as

$$N_p = \frac{1}{p!} (\xi^p - b) \quad \begin{aligned} b &= 1 \quad \text{if } p \text{ even} \\ b &= \xi \quad \text{if } p \text{ odd} \end{aligned} \quad (2.10)$$

where  $p \geq 2$  is the polynomial order.

It is easy to observe now the associated variables have the meaning of higher derivatives of  $\phi$  i.e.,

$$a_{p+1} = \frac{d^p \hat{\phi}}{d\xi^p} \quad (2.11)$$

Although the successive importance of these hierarchical variables diminishes, the optimal form is one which gives an orthogonal set of shape functions in relation to the energy inner product. Such a set of shape functions will be given by integrals of the Legendre polynomials in the following form

$$N_p = \frac{1}{(p-1)!} \frac{1}{2^{p-2}} \frac{d^{p-2}}{d\xi^{p-2}} (1-\xi^2)^{p-1} \quad (2.12)$$

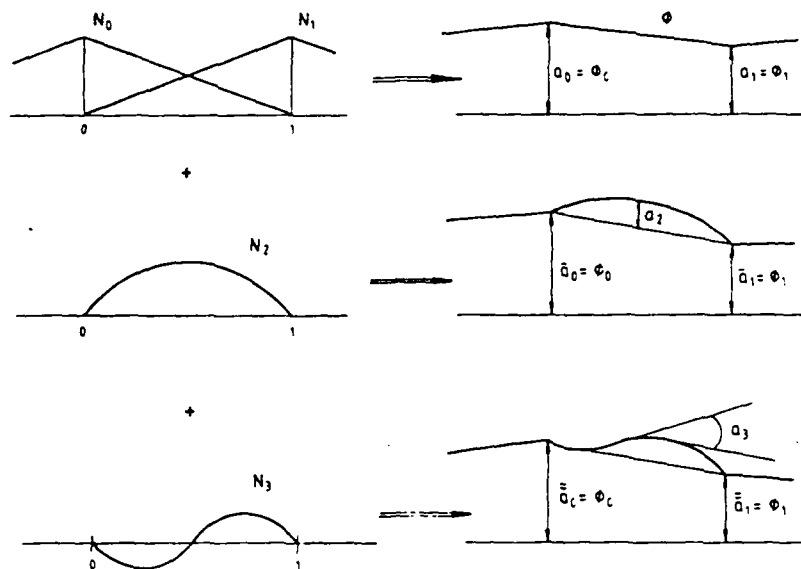


FIG. 1. One-dimensional hierarchical elements for the p-version of the finite element method

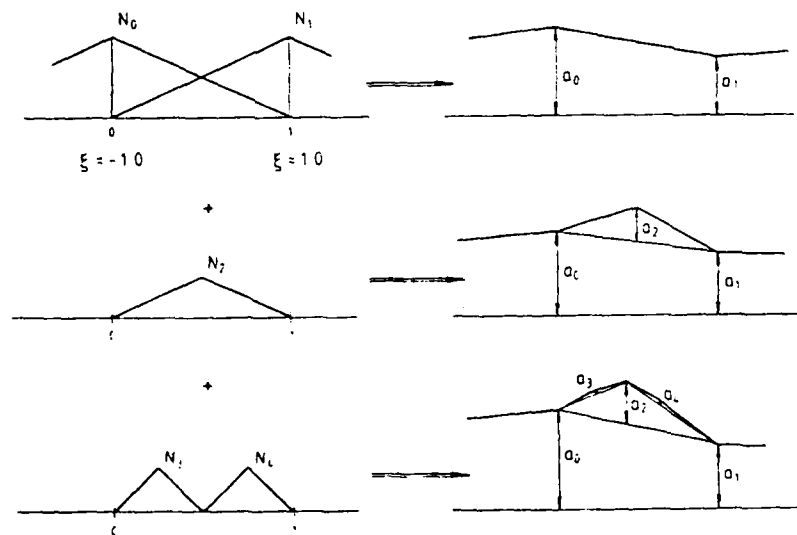


FIG. 2. One-dimensional hierarchical elements for the h-version of the finite element method

if the differential equation  $-\phi'' + q = 0$  is considered.

This set was introduced by Zienkiewicz et al. [58]. Again, the multipliers have the meaning of a measure of the higher derivatives of  $\phi$  at the centre of the finite element. The advantage of this set of hierarchical functions is that for 1-D problems the coupling between different higher order degrees of freedom is non-existent. We will discuss this point further in Section 2.2. We shall also see that this set of hierarchical functions will play a very important role in the error analysis study.

For the h-version there is also an infinite number of hierarchical refinement possibilities corresponding to the sub-division of the initial element in equal or unequal parts. If we consider a hierarchical refinement corresponding to Figure 2, we will have as shape functions

$$\begin{aligned} N_0 &= \frac{1-\xi}{2} \\ N_1 &= \frac{1+\xi}{2} \\ N_2 &= \begin{cases} \xi + 1 & \text{if } \xi \leq 0 \\ -\xi + 1 & \text{if } \xi \geq 0 \end{cases} \end{aligned} \quad (2.13)$$

etc. Considering Figure 2 it is obvious that the physical meaning of the linear hierarchical variables is a relative displacement set.

Once the one-dimensional interpolation formulae have been established the generation of hierarchical shape functions for rectangular elements is almost trivial as,

- a) the corner node functions are simply bi-linear products, and
- b) 'hierarchical' functions of the type defined above are always zero at the corner nodes.

Polynomial shape functions of all orders in two dimensions can be obtained by simple products of the one-dimensional formulae, but in general losing the properties of the orthogonality mentioned above. The identity of hierarchical variables on any element side with those on the adjacent element then automatically guarantees the uniqueness of the polynomial along that side.

The three-dimensional case and the triangular based finite elements are just special cases of the concepts expanded above and we direct the interested reader to [36, 62].

The hierarchical and non-hierarchical shape functions for the h and p versions of the finite element method are presented in Figure 3 and Figure 4 for two-dimensional problems. It is easy to show that a direct transformation from the hierarchical to the non-hierarchical formulation and vice-versa is possible. For example, it is possible to transform a quadratic serendipity



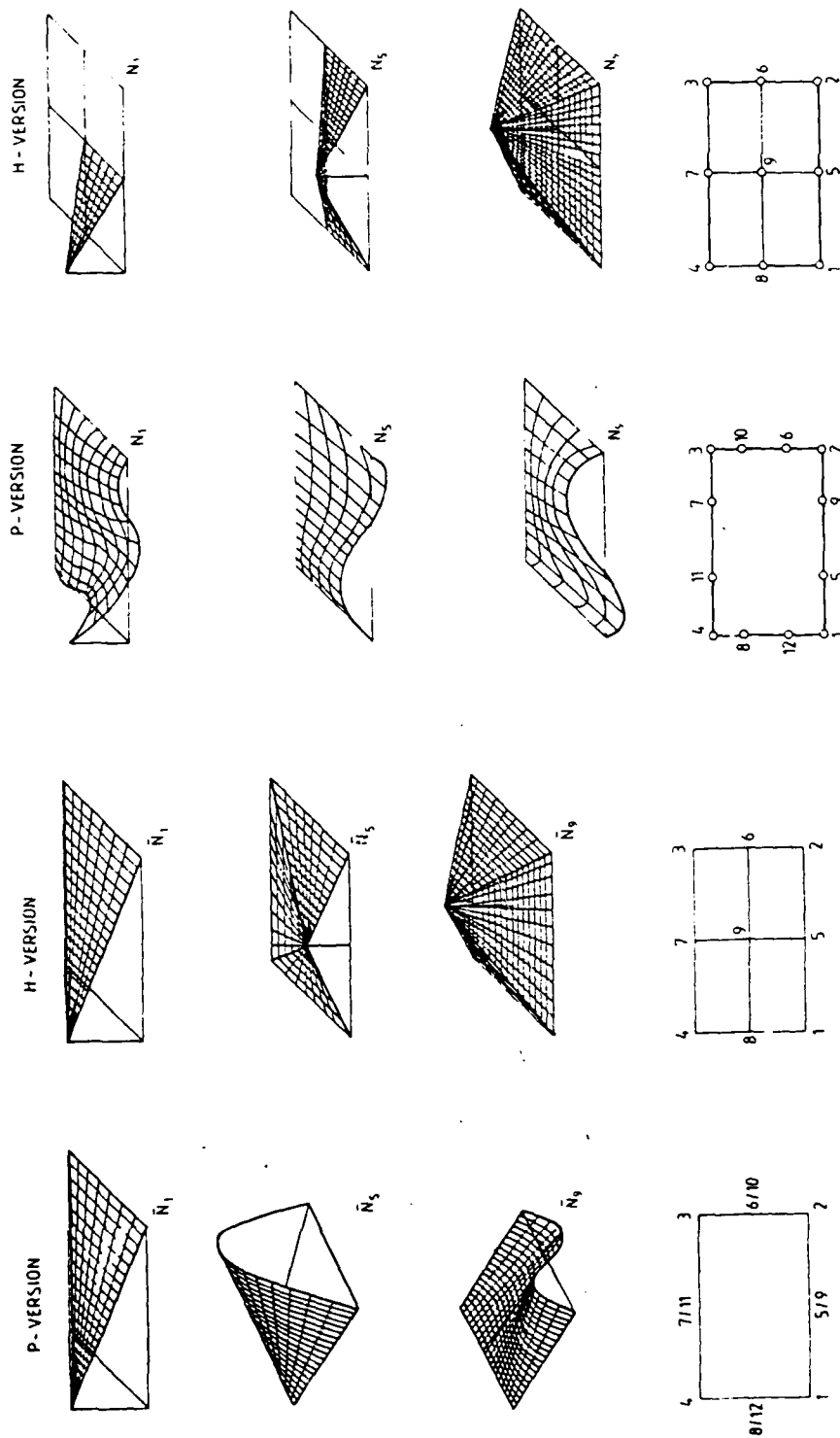


FIG. 3. Two dimensional hierarchical shape functions for the p and h-versions of the finite element method

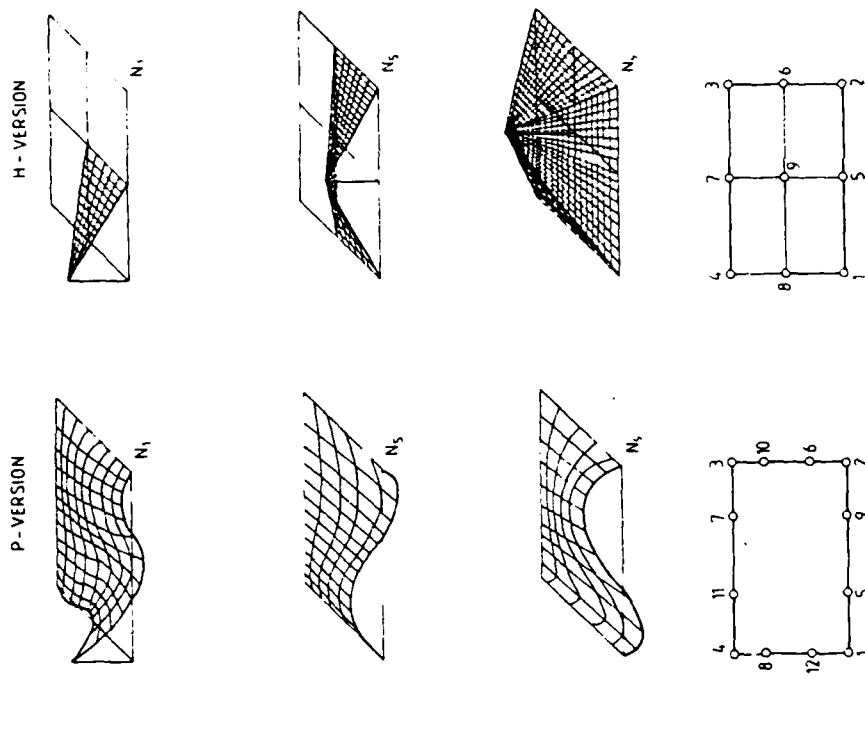


FIG. 4. Two dimensional non-hierarchical shape functions for the p and h-versions of the finite element method

element to hierarchical form by a matrix transformation after the non-hierarchical matrix has been determined. It should also be noted that when the refinement is to the same order in Figures 1 to 4, the finite element solution  $\hat{\phi}$  in (2.5) from the hierarchic and non-hierarchic formulations will be identical.

## 2.2 Advantages of Hierarchical Formulations

In 1-D the advantages of a hierarchical formulation over a non-hierarchical one are obvious because we can obtain a system of equations of the form

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & \cdot & 0 \\ K_{21} & K_{22} & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & K_{33} & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & K_{44} & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & K_{55} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & K_{nn} \end{bmatrix} \begin{Bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \\ \underline{a}_4 \\ \underline{a}_5 \\ \vdots \\ \underline{a}_n \end{Bmatrix} + \begin{Bmatrix} \underline{q}_1 \\ \underline{q}_2 \\ \underline{q}_3 \\ \underline{q}_4 \\ \underline{q}_5 \\ \cdot \\ \underline{q}_n \end{Bmatrix} = \underline{0} \quad (2.14)$$

where  $K_{ii}$  are diagonal matrices if the functions are of a suitable orthogonal form. This implies an improved conditioning of the assembled system of equilibrium equations, compared to the non-hierarchical formulation which does not have the same strong diagonal character and, secondly, the possibility of a direct solution for the hierarchical variables.

In 2-D we expect the same advantages to hold although the system of equations is now not completely orthogonal and has the form

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \underline{a}_1 \\ \underline{a}_2 \end{Bmatrix} + \begin{Bmatrix} \underline{q}_1 \\ \underline{q}_2 \end{Bmatrix} = \underline{0} \quad (2.15)$$

The first advantage noted above is carried to the two-dimensional case by the fact that for each element the conditioning of the hierarchical stiffness matrix is better than the conditioning of the non-hierarchical one. The off-diagonal links are weakened, implying an overall better conditioning of the resulting system of equations. This is in fact observed by numerous authors [30, 53-56], who have proved that a relative displacement formulation of the finite element method yields better conditioned matrices than the classic total variable approach. Indeed, the same can be said of so-called local-global element forms.

In relation to the second advantage we can say that the hierarchical formulation is optimal because it allows for all the information to be passed from one discretization level to the second discretization level once a mesh refinement is decided. Also the implicit substructuring existent in this multi-level formulation allows for the very effective use of block iteration solution schemes. This, associated with the better conditioning of the overall system, will imply a fast rate of convergence for the iterative equation solver. Wachspress two-level elements [54] are another way of achieving these objectives although on a two-level theory, rather than a multi-level one. See also the local-global formulation of Mote [30] as another possible form of hierarchical formulation.

### 3. ERROR ANALYSIS (A HIERARCHICAL APPROACH)

The function  $\hat{\phi}$  of the form (2.5) is not in general the exact solution of the problem. Therefore we cannot, in general, have

$$L \hat{\phi} + q = r \equiv 0 \quad (3.1)$$

$$\bar{L} \hat{\phi} + s = \rho \equiv 0 \quad (3.2)$$

(if (3.1) (3.2) would be satisfied, we would have  $\hat{\phi} = \phi$ ).

Assume for simplicity that (3.2) (i.e.  $\rho = 0$ ) holds; then  $r \neq 0$ . Function  $r$  usually (e.g. if  $L$  is a second-order equation and  $N_i$  have not continuous derivative) can be written in the form

$$r = r_1 + r_2 \quad (3.3)$$

where  $r_1$  is the (usual) regular part of the residuum inside every element and  $r_2$  is the singular part (Dirac) function concentrated on the interface between elements with physical interpretation of concentrated forces whose origin is indicated in Figure 5.

Considering e.g. equation

$$-\phi'' + q = 0 \quad \phi(0) = \phi(L) = 0 \quad (3.4)$$

and using piecewise linear elements (as in Figure 5) we get the regular part

$$r_1 = q$$

and the singular part  $r_2$  are the forces (Dirac function) concentrated at the nodal point with the magnitude of the jump of the derivative of the approximate solution on the given nodal point.

For the potential problem in two dimensions the Dirac function is concentrated on the element interface and has the magnitude

$$J_s = \frac{\partial \hat{\phi}}{\partial n}_1 - \frac{\partial \hat{\phi}}{\partial n}_2$$

where  $\frac{\partial \hat{\phi}}{\partial n}_i$ ,  $i=1,2$  is the normal derivative on the left and right side of the interface. In the case of elasticity,  $J_s$  is the traction discontinuity between elements.

### 3.1 Error Definition in the Energy Norm

Denoting  $e = \phi - \hat{\phi}$ , the error of the finite element solution, then obviously subtracting (2.3) (3.1) and (2.4) (3.2) we get

$$Le = -r$$

$$\bar{L}e = -\rho \quad (3.5)$$

and we assume for simplicity as before that  $\rho = 0$ . The goal is now to measure the magnitude of  $e$ . If the set of the trial functions and test functions is the same, then for a suitable class of linear problems we can define

$$\|e\|_E = \left[ \int e Le d\Omega \right]^{\frac{1}{2}} = \left[ \int (\phi - \hat{\phi}) L(\phi - \hat{\phi}) d\Omega \right]^{\frac{1}{2}} \quad (3.6)$$

the so-called energy norm, and through it we can measure the magnitude of  $e$ .

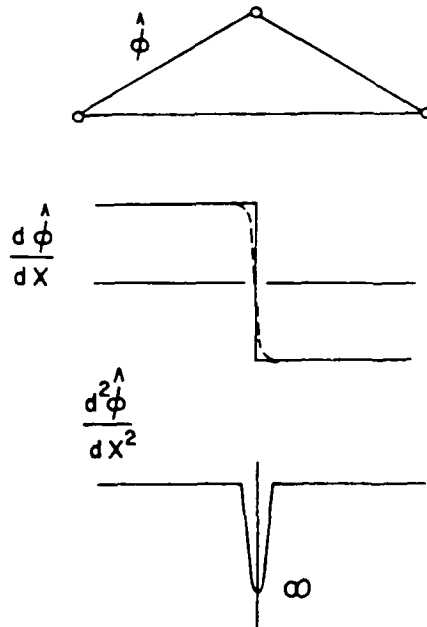


FIG.5. Interface residuals for error analysis.

Expanding  $||e||_E^2$  we have

$$||e||_E^2 = \int \phi L\phi d\Omega + \int \hat{\phi} L\hat{\phi} d\Omega - \int \phi L\hat{\phi} d\Omega - \int \hat{\phi} L\phi d\Omega \quad (3.7)$$

Note that by equation (2.6) with  $W_e = \hat{\phi}$  we can write

$$\int \hat{\phi} L\hat{\phi} d\Omega \equiv - \int \hat{\phi} q d\Omega \quad (3.8)$$

and by equation (2.3)  $L\phi = -q$ . Substituting in (3.7),

$$||e||_E^2 = - \int \phi (L\hat{\phi} + q) d\Omega \quad (3.9)$$

$$\text{or} \quad ||e||_E^2 = - \int \phi r d\Omega \quad (3.10)$$

Using once more (2.6) and (3.8) we get

$$\int \hat{\phi} r d\Omega = 0 \quad (3.11)$$

so we can write

$$||e||_E^2 = - \int (\phi - \hat{\phi}) r d\Omega \quad (3.12)$$

$$\text{or} \quad ||e||_E^2 = - \int e r d\Omega \quad \text{where } e = \phi - \hat{\phi} \quad (3.13)$$

We remark that (3.11) has a sense of a self equilibration of the residual forces.

### 3.2 Error Indication in One Dimension

We now come to the crux of the matter. How can we estimate the error using expressions such as (3.12) without knowledge of  $\phi$ ?

It is clear that what is needed is an approximation to  $\phi$  (or  $e$ ) on a local base because St.Venant's Principle ensures the effect of the equilibrating residuals will be local.

Employing hierarchical modes provides one possibility. On every element  $I_j$  we set

$$\phi \approx \hat{\phi} = \hat{\phi} + N_{i+1} a_{i+1} \quad (3.14)$$

where  $N_{i+1}$  is the (as yet not used) next hierarchical shape function. Now we could write

$$\|e\|_{E(I_j)}^2 \approx -a_{i+1} \int N_{i+1} r d\Omega \quad (3.15)$$

where  $\|\cdot\|_{E(I_j)}$  is the norm of the error of the element  $I_j$ .

This process obviously needs an estimate of  $a_{i+1}$ .

In one dimension, when we choose as shape functions polynomials whose derivatives are orthogonal, we have from (2.14)

$$a_{i+1} = -\frac{q_{i+1}}{K_{i+1,i+1}} \quad (3.16)$$

Here  $q_{i+1}$  can be determined from

$$q_{i+1} = \int N_{i+1} q d\Omega = \int N_{i+1} (r - L\hat{\phi}) d\Omega = \int N_{i+1} r d\Omega \quad (3.17)$$

because  $\int N_{i+1} L\hat{\phi} d\Omega = 0$  due to the orthogonality referred to above. Now (3.11) can be written as

$$\|e\|_{E(I_j)}^2 \approx \frac{(q_{i+1})^2}{K_{i+1,i+1}} \quad (3.18)$$

which is the error indicator presented in [37,40].

When the shape functions do not possess a complete orthogonality a degree of approximation has to be introduced here.

First,  $a_{i+1}$  has to be estimated using the previously found  $a_j$  values as

$$a_{i+1} = \frac{1}{K_{i+1,i+1}} (-q_{i+1} - K_{i+1,j} a_j) \quad (3.19)$$

Now

$$q_{i+1} = \int N_{i+1} r d\Omega - K_{i+1,j} a_j \quad (3.20)$$

because

$$\int N_{i+1} L\phi_i d\Omega \neq 0 \quad (3.21)$$

and  $q_{i+1}$  is again evaluated from the previously determined  $a_j$ . These approximations are generally tenable if near orthogonality of hierarchical functions exists, as is often the case with the p-type elements.

A more serious shortcoming of this error estimate is however immediately apparent. If the residual  $r$  is orthogonal to  $N_{i+1}$  in (3.15) the error indication will be zero, since

$$\int N_{i+1} r \, d\Omega = 0 \quad (3.22)$$

This implies that the proposed criterion leads to an indication of the error absorbed by a hierarchical refinement on the existing mesh rather than an accurate estimate of the error in the finite element solution. Therefore we will denote this simply as an error indicator and search further for a true error "estimate".

### 3.3 Error Estimation in One Dimension

Such an estimate can be obtained from a simple calculation, when  $e = \phi - \hat{\phi}$  is the exact response to  $r$ .

Consider the model problem

$$-\frac{d^2\phi}{dx^2} + q = 0 \quad (3.23)$$

with boundary condition

$$\phi(0) = \phi(L) = 0$$

with  $e$  being the error involved in a linear approximation with nodal points  $x_i$ . As in (3.5), we have

$$-\frac{d^2e}{dx^2} + r = 0 \quad \text{with } e(0) = e(L) = 0 \quad (3.24)$$

In addition it can be shown that  $e(x_i) = 0$  in our particular case (e.g. [59]), so  $e$  can be determined on every interval  $I_j = (x_{j-1}, x_j)$  separately.

Assuming that  $r = \bar{r} \sin \frac{\pi x}{L}$ , then the differential equation

$$-\frac{d^2e}{dx^2} + r = 0 \quad e(0) = e(L) = 0 \quad (3.25)$$

is easily solvable. Substituting  $e = \bar{e} \sin \frac{\pi x}{L}$  in (3.25) we get

$$\frac{\pi^2}{L^2} \bar{e} \sin \frac{\pi x}{L} = -\bar{r} \sin \frac{\pi x}{L} \quad (3.26)$$

$$\text{giving } e = -\frac{L^2}{\pi^2} r \quad (3.27)$$

Further on  $L$

$$\|e\|_E^2 = - \int_0^L e r dx = \frac{L^2}{\pi^2} \int_0^L r^2 dx \quad (3.28)$$

So far we assumed that  $r = \bar{r} \sin \frac{\pi x}{L}$ . In general, the residual will involve many terms and

$$e = \sum a_i \sin \frac{i\pi x}{L}.$$

It can be easily shown that we get (see Appendix)

$$\|e\|_E^2 \leq \frac{L^2}{\pi^2} \int_0^L r^2 dx \quad (3.29)$$

i.e., we replace equality in (3.28) with inequality in (3.29). Applying now (3.28) to every interval separately [because  $e(x_i)=0$ , it is possible] then we get

$$\|e\|_E^2 \leq \frac{1}{\pi^2} \sum (x_{i+1} - x_i)^2 \int_{x_i}^{x_{i+1}} r^2 dx \quad (3.30)$$

where  $x_{i+1} - x_i = h_{i+1}$ .

The bounding inequality in (3.30) is valid for  $r$  of all orders of polynomial variation in the one-dimensional problem (3.23) and linear elements. Note, however, that in the limit of  $h$  refinement of the linear elements we expect that  $r$  will be nearly constant on each element. In this limit only, the addition of a hierarchic quadratic term will give a valid error estimate. Here we can rewrite (3.14) on every single element  $I$  of length  $h$

$$\phi = \hat{\phi} + x(h-x) a_{i+1} \quad (3.31)$$

and

$$e = a_{i+1} x(h-x) \quad (3.32)$$

From (3.24)

$$r = -2a_{i+1} \quad (3.33)$$

and substituting (3.33) in (3.32)

$$e = -\frac{r}{2} x(h-x) \quad (3.34)$$

Then

$$\|e\|_{E(I)}^2 = - \int_0^h e r dx = \frac{r^2}{2} \int_0^h x(h-x) dx = \frac{r^2 h^3}{12} \quad (3.35)$$



Note that (3.32) will give the same measure of the error for constant  $r$  if the factor  $\pi^2$  is changed to 12. We therefore implement

$$\|e\|_E^2 = \frac{1}{12} \sum_i h_i^2 \int_{x_i}^{x_{i+1}} r^2 dx \quad (3.36)$$

as the asymptotically correct error estimator for linear elements and one-dimensional problems. This is indeed the form presented by Babuška et al. [4-6] where a different and detailed mathematical justification is given.

### 3.4 Error Indication and Estimation in Two Dimensions

In two dimensions the hierarchical functions have their support on either one or two elements, as shown in Figure 6. Again we can use the hierarchical modes to make a total projection as in (3.15). We note, however, that in two dimensions the hierarchical contributions to  $K$  are not diagonal as in the 1-D equation, and the approximation of (3.19) has to be used, i.e.

$$a_{i+1} = - \frac{[q_{i+1} + K_{i+1,j} a_j]}{K_{i+1,i+1}} \quad (3.37)$$

(for  $j \neq i+1$  and summation convention) then

$$\left( \|e\|_E^2 \right)_{i+1} \approx \frac{[q_{i+1} + K_{i+1,j} a_j]^2}{K_{i+1,i+1}} \quad (3.38)$$

where the subscript  $i+1$  refers to the new hierarchic mode.

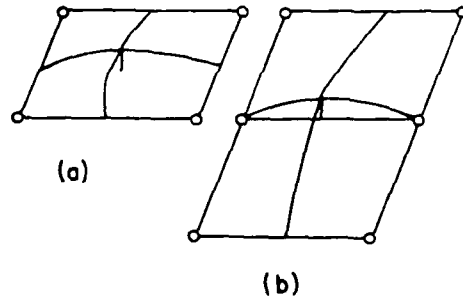


FIG.6. Hierarchical supports for two-dimensional error analysis.

Note that if we sample each hierarchical mode independently (set other hierarchical amplitudes  $a_i = 0$ ) then the interaction of the new modes in the solution is ignored, introducing a new approximation in addition to the problem of this error indicator being a projection of the true error in the next hierarchical mode and not the total error. The effect of this new approximation cannot be evaluated a-priori.

To estimate the total error we will take some guidance from the hierarchic nodes in Figure 6 but attempt to base the analysis on the theory developed for the one-dimensional case. The hierarchic modes indicate that not only must we consider the contribution from the residual on the element (regular part of the residual) but also the contribution from the interface (singular part of the residual). In one dimension only the former was considered because the hierarchic nodes were all internal to the element.

Consider first the regular part of the residual on the element. We will over-estimate the error associated with the hierarchic node in Figure 6a by releasing boundary conditions to give two one-dimensional responses (see Figure 7a) and associating half the residual in each direction. In each direction (3.30) gives

$$\|e\|_{E(I_j)}^2 \leq \frac{h_j^2}{\pi^2} \int \left(\frac{r}{2}\right)^2 d\Omega \quad (3.39)$$

so that the addition of both one-dimensional contributions gives

$$\|e\|_{E_1}^2 \leq \frac{h^2}{2\pi^2} \int_{\Omega_j} r^2 d\Omega \quad (3.40)$$

The influence of the singular part of the residual can be treated in the same way if the residual is distributed as indicated in Figure 7b. Now we have

$$r \approx \frac{J_s}{h} \quad (3.41)$$

and (3.30) gives

$$\begin{aligned} \|e\|_{E_2}^2 &\leq \frac{h^2}{\pi^2} \int r^2 d\Omega \\ &= \frac{h^2}{\pi^2} \int \left(\frac{J_s}{h}\right)^2 d\Omega \\ &= \frac{h}{\pi^2} \int J_s^2 dy \end{aligned} \quad (3.42)$$

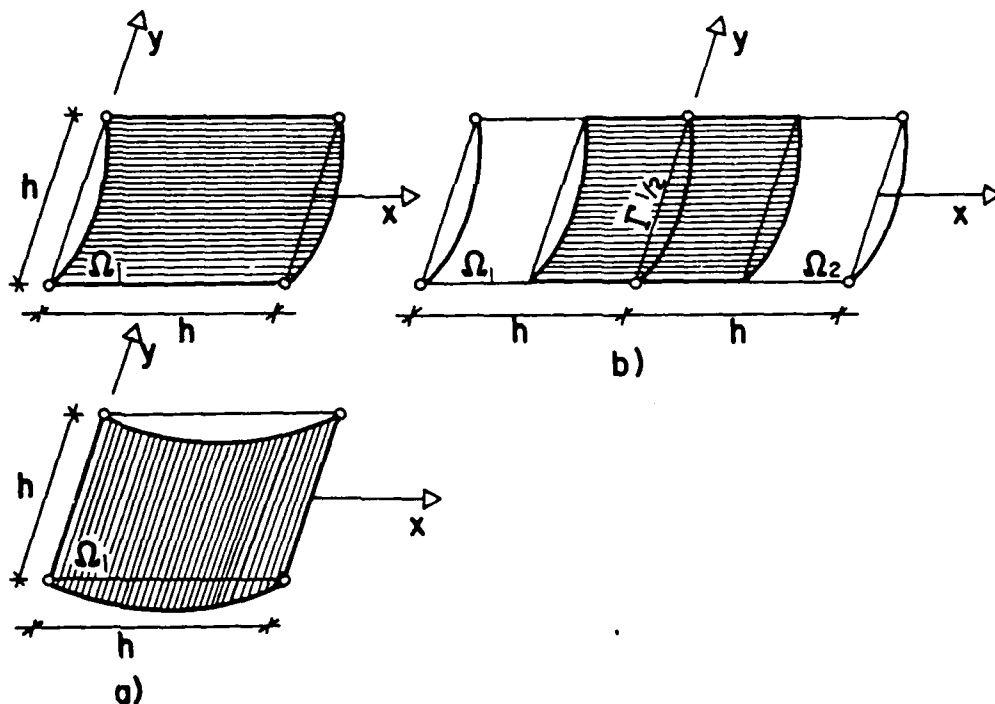


FIG.7. Deformation patterns for two-dimensional error analysis.

The error estimator will consider each element separately. We introduce a factor of  $1/2$  into (3.42) because each boundary will be considered twice. The error estimate becomes for the bi-linear element

$$\|e\|_{E(\Omega_j)}^2 \leq \frac{h^2}{2\pi^2} \int_{\Omega_j} r^2 d\Omega + \frac{h}{2\pi^2} \int_{\Gamma_j} J_s^2 d\Gamma \quad (3.43)$$

Since this derivation is based on a superposition of one-dimensional analyses we should obtain the one-dimensional estimate if the boundary conditions impose essentially a one-dimensional problem on the two-dimensional mesh. Using (3.43) and constant right-hand side  $q$  in (2.3) we get an estimate which is not completely identical with the one-dimensional estimate. Therefore we reduce the estimate by modifying (3.43) to

$$\|e\|_{E(\Omega_j)}^2 \leq \frac{h^2}{2\pi^2} \int_{\Omega_j} (r - \bar{r})^2 d\Omega + \frac{h}{2\pi^2} \int_{\Gamma_j} J_s^2 d\Gamma \quad (3.44)$$

where  $\bar{r}$  is the mean value of the residual on the element. This form is mathematically justified in [2].

It follows that on the entire domain we get

$$\|e\|_E^2 \leq \sum_i \left( \frac{h^2}{2\pi^2} \int_{\Omega_i} (r - \bar{r})^2 d\Omega + \frac{h}{2\pi^2} \int_{\Gamma_i} J_s^2 d\Gamma \right) \quad (3.45)$$

where the sum is taken over all elements  $\Omega_i$  and  $\Gamma_i$  (if  $J_s$  at the boundary  $\Gamma_i$  is properly defined).

We derived these estimates under various assumptions. The question arises whether they are acceptable. The answer is positive (see for example Babuska et al. [2,7]). There the term  $1/\pi^2$  is replaced by  $1/12$  to obtain the asymptotically correct estimate

$$\|e\|_E^2 \approx \sum_i \left( \frac{h^2}{24} \int_{\Omega_i} (r - \bar{r})^2 d\Omega + \frac{h}{24} \int_{\Gamma_i} J_s^2 d\Gamma \right) \quad (3.46)$$

The estimate has essentially two parts. The one related to the regular part of residual (volume integral) and the other one to the singular part of the residual (jump of derivatives term). It can be theoretically shown that the first term (volume integral) is in the limit negligible with respect to the second one. Practical experience shows that the first term is relatively small also for coarse meshes.

The estimate is called asymptotically correct when the ratio of the right and left-hand side of (3.46) goes to one as the error goes to zero. The estimate (3.46) has this property when some mathematical assumptions are satisfied. One major one is that the element error estimators are about the same in magnitude. This can be achieved e.g. by an adaptive selection of the elements. The experience shows e.g. that the asymptotical correctness is not achieved when the solution has singular behaviour and a uniform (obviously improper) mesh is used.

### 3.5 Practical Error Analysis

Both the estimators (3.36) and (3.43) and the indicators (3.18) and (3.38) can play a fundamental role in the finite element analysis. The estimators allow for an evaluation of the total error in energy in the current finite element solution and the indicators allow a rational increase of degree of the element (made in a hierarchical way) and/or element subdivision.

The latter is a direct consequence of the hierarchical error indicators being a projection of the error in the new hierarchical modes, reflecting thus the capacity for the new modes to 'absorb' error in energy. In addition, the possibility of obtaining an accurate estimate of the error in an appropriate norm allows the program to stop automatically when a certain accuracy

has been achieved or to indicate the order of accuracy when a certain pre-specified solution cost has been attained.

It seems, however, that to obtain both advantages we have not only to compute all the residuals and stress discontinuities at every stage of the iteration process to evaluate the estimate (3.46), but also all the hierarchical stiffness coefficients corresponding to the possible new refinements to evaluate the indicator (3.38). This is not necessarily the case, since the error indicators can be obtained as projections of the computed residuals and stress jumps in the new hierarchical modes, i.e. in (3.38)

$$q_j + K_{ji} a_i$$

$$= \int_{\Omega_1} r_1 N_j d\Omega + \int_{\Omega_2} r_2 N_j d\Omega - \int_{\Gamma_{1/2}} J_s N_j d\Gamma$$

where  $\Omega_1, \Omega_2$ , and  $\Gamma_{1/2}$  and  $J_s$ , are indicated in Figure 7b. This computation can be achieved locally.

**3.5.1 Requirements for Practical Error Estimates.** We consider that a practical error estimator should satisfy the following conditions:

- a. Be determined a-posteriori from information defined on a local basis.
- b. If we define an effectivity index  $\theta = \frac{\|e\|_E}{\left(\|e\|_E\right)_{\text{exact}}} \quad (\text{i.e.})$   
the ratio of the predicted energy norm of the error to the exact value of this norm), then we require  $\theta \geq 1$  for all meshes, and to provide reasonable bounds  $1 \leq \theta \leq 2$ .
- c. Asymptotic convergence  $\theta \rightarrow 1$ .
- d. A direct interpretation of errors in stresses should be available.

In the examples that follow we show that error estimates which satisfy these criteria are available from the estimators of the form (3.36) and (3.46) if the following amendments are incorporated:

1. Theorem 2 of [7] states that there exist constants  $k_2 \geq k_1 > 0$  independent of the mesh, such that

$$k_1 \|e\|_E \leq \left(\|e\|_E\right)_{\text{exact}} \leq k_2 \|e\|_E$$

with  $\|e\|_E$  given by (3.46). Experience has shown  $k_1 \approx 0.5$  and  $k_2 \leq 1.5$  on a large number of problems, and both are asymptoti-

cally equal to 1 for uniform meshes and smooth solutions. To prevent gross violation of the second condition required of the error estimates we seek a value of  $k_2$  by defining a factor  $k_{2i}$  for element  $i$  as

$$k_{2i} = 1 + \alpha \frac{\|e\|_{E(\Omega_i)}}{\|u_h\|_{E(\Omega_i)}} \quad (3.47)$$

where

$$\|u_h\|_{E(\Omega_i)} = \left( u_h^T K^i u_h \right)^{\frac{1}{2}} \quad (3.48)$$

is the energy on the  $i$ -th finite element. The corrected error estimate becomes

$$\|e\|_{E(\Omega_i)}^* = k_{2i} \|e\|_{E(\Omega_i)} \quad (3.49)$$

We have taken  $\alpha = 2$  in all applications.

2. For isoparametric transformation of the element there is more than one possible choice for the length parameter  $h$ . Following [19] and because we seek an over-estimate of the error in the solution, we choose  $h$  as the length of the maximum side of the element.

3. Error estimators have not been developed for elements of higher order than linear. However, significant work is being done in this area by Szabo and his co-workers [46-49].

The sine function analysis affords the following extension.

Quadratic elements would match the predominant part of the error in the first sine function  $i=1$ , used for linear elements. Assuming now the error in the form of the sine function with  $i=2$  we would replace the coefficient

$$\frac{h^2}{2\pi^2} \quad \text{by} \quad \frac{h^2}{n\pi^2} \quad (3.50)$$

with  $n=8$ . For the cubic element we could progress to the mode  $i=3$  to get the coefficient with  $n=18$ .

Thus with changing polynomial order  $p$  on the elements the coefficient of the estimator can be replaced by

$$\frac{h^2}{2\pi^2} \left( \frac{1}{p^2} \right)$$

It is to be noted however, that this assumes that quadratic

interpolants, for example, can completely eliminate error in the form of the first sine function. This of course is not the case so the factor above would be optimistic.

Again, in the interest of producing an over-estimate of the error in the solution, we reduce the power on  $p$  to one so that a general form of (3.46) for elements of polynomial order  $p$  becomes

$$\|e\|_{E(\Omega_i)}^2 = \frac{h^2}{24p} \int_{\Omega_i} r^2 d\Omega + \frac{h}{24p} \int_{\Gamma_i} J_s^2 d\Gamma \quad (3.51)$$

Note we relax the requirement  $\theta \rightarrow 1$  here in 2. above.

4. The analysis above strictly applies only to the Laplace operator. However, it can be generalized, for example, to problems governed by the Navier equations of elasticity. Formula (3.51) has been found adequate when the influence of Poisson's ratio has been incorporated. We take

$$\|e\|_{E(\Omega_i)}^2 = C \left\{ \frac{h^2}{24p} \int_{\Omega_i} r^2 d\Omega + \frac{h}{24p} \int_{\Gamma_i} J_s^2 d\Gamma \right\} \quad (3.52)$$

with  $C = \frac{1-\nu}{E}$  for plane stress.

5. Local estimates of the error in stress are the elusive goal of most practical error analysis. It is interesting that the error estimates advocated here are evaluated locally element-wise are justified locally by appealing to St. Venant's Principle and are backed up by a corrective factor which, in a limited number of experimental problems, ensured local bounds on the energy of each element.

We can suggest the following: a bound on some average of the stress on the element will be obtained by scaling local stresses by a factor  $FS_i$

$$FS_i = \frac{\|u_e\|'_{E(\Omega_i)}}{\|u_h\|_{E(\Omega_i)}} \quad (3.53)$$

where  $\|u_e\|'_{E(\Omega_i)}$  is the prediction of the energy of the exact solution on the  $i$ -th element evaluated as

$$\|u_e\|'_{E(\Omega_i)} = \left( \|u_h\|_{E(\Omega_i)}^2 + \|e\|_{E(\Omega_i)}^2 \right)^{\frac{1}{2}} \quad (3.54)$$

and  $\|u_h\|_{E(\Omega_i)}$  is given by (3.48).

### 3.6 Applications of the Error Estimators of Section 3.5.

*3.6.1 Cantilever with applied end moment, linear elements.* The simple cantilever shown in Figure 8 was analysed for plane stress using the four-node, bilinear element. The exact solution has a quadratic variation of vertical displacement along the beam.

The finite elements used are compatible and fully integrated so the finite element solution  $u_h$  is known to be stiff. The error estimate  $\|e\|_E^*$  does, however, over-estimate the error in the solution. A comparison of the results also supports the constant  $C$  in (3.52) used to include the influence of Poisson's ratio  $\nu$ . Finally, the moment is constant along the beam in this simple example so that the stresses in each element are identical and  $\|e\|_{E(\Omega_i)}$  from (3.52) is the same for each element. The

error estimates in the table can therefore be interpreted locally as well as globally, and implementation of (3.53) will obviously give good estimates of the local error in stress.

The last two columns in the table give the ratios

$$\frac{\|e\|_E^*}{\|u_h\|_E} \times 100 \quad \text{and} \quad \frac{\left(\|e\|_E\right)_{\text{exact}}}{\|u_h\|_E} \times 100$$

respectively. This measure may provide the best qualitative indication of the accuracy of the finite element stresses.

Note that all calculations between values in these tables must be based on the formula

$$\left(\|u\|_E^2\right)_{\text{exact}} = \|u_h\|_E^2 + \|e\|_E^2$$

This formula expresses the fact that the error is orthogonal to the finite element solution.

*3.6.2. Small circular hole,  $h$  and  $p$  refinement.* The configuration of the problem and the finite element meshes used are shown in Figure 9. The analysis was for plane stress with  $\nu = 0.3$ . The results given as Case 1 in Table 2 are for the bilinear four-node element and subdivision of all elements into four, as shown in the figure. In this analysis the surface geometry of the hole was updated in the refinement. Again, the global error estimates  $\|e\|_E^*$  are excellent.



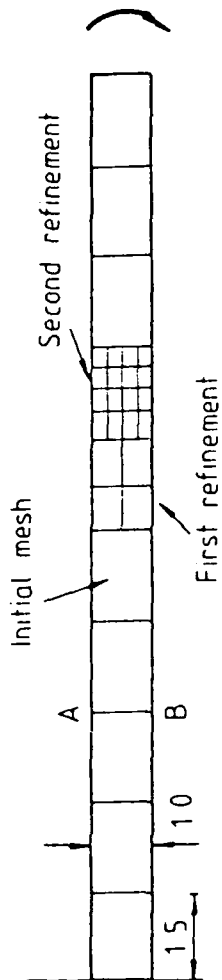


Fig. 8. Cantilever and adaptive meshes.

TABLE 1

Cantilever Applied End Moment, Linear Elements

| (a) $\nu = 0.0$  |                                     | Error Estimate        |      | Error Estimate        |      | $\frac{\ e\ _E^* . 100}{\ u_h\ _E}$ |     | $\frac{(\ e\ _E)_{\text{exact}} . 100}{\ u_h\ _E}$ |  |
|------------------|-------------------------------------|-----------------------|------|-----------------------|------|-------------------------------------|-----|--|--|
| No. Elements     | $\ u_h\ _E$                         | $\ e\ _E$             | 0    | $\ e\ _E^*$           | 0*   |                                     |     |  |  |
| 10               | $.153 \times 10^{-2}$               | $.101 \times 10^{-2}$ | 0.62 | $.233 \times 10^{-2}$ | 1.43 | 152                                 | 107 |  |  |
| 40               | $.198 \times 10^{-2}$               | $.082 \times 10^{-2}$ | 0.78 | $.150 \times 10^{-2}$ | 1.43 | 75                                  | 53  |  |  |
| 160              | $.216 \times 10^{-2}$               | $.049 \times 10^{-2}$ | 0.85 | $.078 \times 10^{-2}$ | 1.36 | 36                                  | 27  |  |  |
| "Exact" Solution | $.224 \times 10^{-2}$ (Beam Theory) |                       |      |                       |      |                                     |     |  |  |
| (b) $\nu = 0.3$  |                                     | Error Estimate        |      | Error Estimate        |      | $\frac{\ e\ _E^* . 100}{\ u_h\ _E}$ |     | $\frac{(\ e\ _E)_{\text{exact}} . 100}{\ u_h\ _E}$ |  |
| No. Elements     | $\ u_h\ _E$                         | $\ e\ _E$             | 0    | $\ e\ _E^*$           | 0*   |                                     |     |  |  |
| 10               | $.160 \times 10^{-2}$               | $.094 \times 10^{-2}$ | .60  | $.229 \times 10^{-2}$ | 1.46 | 143                                 | 98  |  |  |
| 40               | $.201 \times 10^{-2}$               | $.076 \times 10^{-2}$ | .72  | $.146 \times 10^{-2}$ | 1.43 | 73                                  | 49  |  |  |
| 160              | $.218 \times 10^{-2}$               | $.045 \times 10^{-2}$ | .84  | $.077 \times 10^{-2}$ | 1.43 | 35                                  | 24  |  |  |
| "Exact" Solution | $.224 \times 10^{-2}$ (Beam Theory) |                       |      |                       |      |                                     |     |  |  |

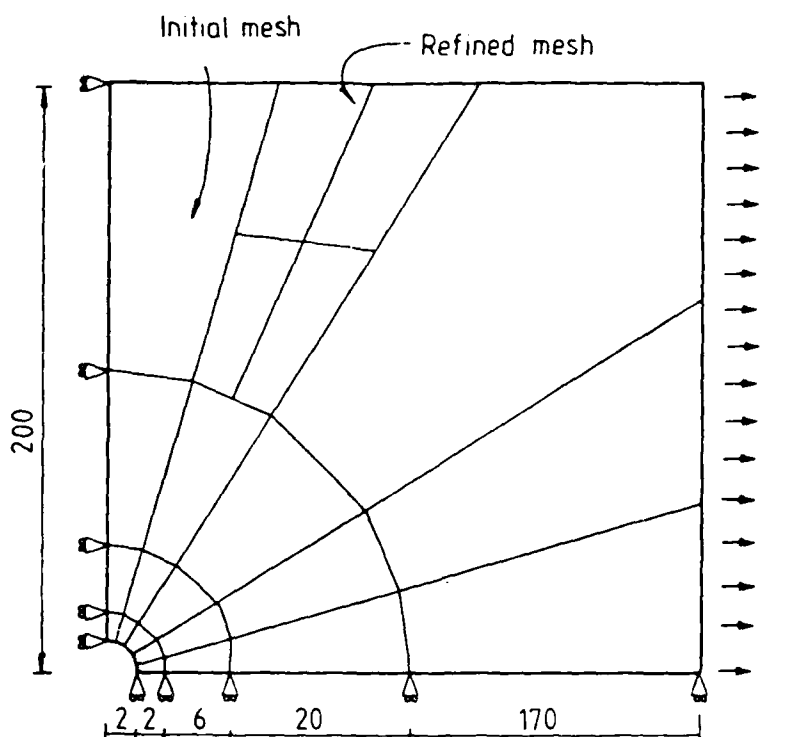


Fig. 9. Small circular hole in large region (not to scale).

TABLE 2

Large Plate with Small Circular Hole  $\nu = 0.3$

Case 1. Linear Elements (Complete Subdivision)

| No. Elements        | $\ u_h\ _E$ | Error Estimate<br>$\ e\ _E$                 | $\theta$ | Error Estimate<br>$\ e\ _E^*$ | $\theta^*$ |
|---------------------|-------------|---|----------|-------------------------------|------------|
| 24                  | 1.154816    | $.583 \times 10^{-2}$                       | .85      | $.700 \times 10^{-2}$         | 1.02       |
| 96                  | 1.154829    | $.343 \times 10^{-2}$                       | .81      | $.405 \times 10^{-2}$         | 0.96       |
| "Exact"<br>Solution | 1.154837    | (refined mesh and Richardson extrapolation) |          |                               |            |

Case 2. Increasing Polynomial Order on 24 Element Mesh (No Update of Hole Surface) (Complete Refinement)

| Element Type (p)    | $\ u_h\ _E$ | Error Estimate<br>$\ e\ _E$                 | $\theta$ | Error Estimate<br>$\ e\ _E^*$ | $\theta^*$ |
|---------------------|-------------|---|----------|-------------------------------|------------|
| Linear (1)          | 1.154816    | $.583 \times 10^{-2}$                       | .85      | $.700 \times 10^{-2}$         | 1.06       |
| Quadratic (2)       | 1.154832    | $.264 \times 10^{-2}$                       | .93      | $.297 \times 10^{-2}$         | 1.05       |
| Cubic (3)           | 1.154834    | $.162 \times 10^{-2}$                       | .94      | $.173 \times 10^{-2}$         | 1.00       |
| "Exact"<br>Solution | 1.154835    | (refined mesh and Richardson extrapolation) |          |                               |            |

The results given as Case 2 in the table are for a uniform increase of the polynomial order on all elements based on the coarse mesh indicated in Figure 9. Here the surface geometry of the hole has not been updated with refinement. Again, the error estimates  $\|e\|_E^*$  are excellent but the example indicates the practical weakness of a global error measure. The first column in the table indicates that the significant error in stresses in the immediate region of the hole surface appears as only a small perturbation to the global energy of the region.

Obviously a local measure is required. In Figure 10 we plot the stress tangential to the hole surface in two elements adjacent to the hole. The finite element stress, plotted as a linear interpolant through the  $2 \times 2$  Gauss point values, is scaled using the factors  $FS_i$  given by (3.55). The order of the error in the stresses is accurately indicated even at this local element level.

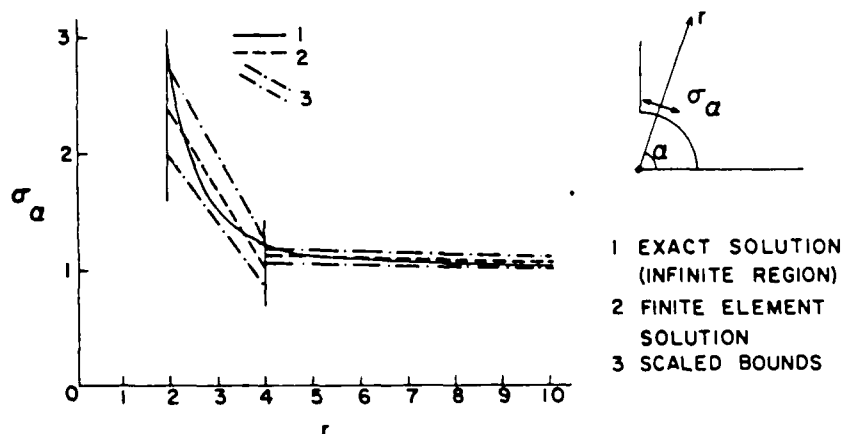


Fig.10. Local stress errors:  $\alpha = 86.4^\circ$  on 24-element mesh.

#### 4. ADAPTIVITY

The concept of adaptivity follows naturally from the previous discussion as the expansion of the trial function space  $S_h$ , hierarchically or non-hierarchically, but only where the space is shown to be deficient. It has been taken as a basis for adaptivity that the sequence of finite element solutions must follow the best rate of convergence in terms of the number of degrees of freedom of the structure. The optimal rates of convergence for both the  $h$  and  $p$  versions of the finite element method have been identified and quantified in [13].

Two programs are being developed for this research. The first is based on the  $h$  convergence process in Babuška [7], which utilizes  $\|e\|_{E(\Omega_i)}$  as the indicator for mesh refinement.

The second p-convergence algorithm uses directly the hierarchical elements and indicators (3.38).

The strategy for selection of new degrees of freedom in adaptive processes is not uniquely resolved. Here we refine on the basis of evaluating all error indicators and including degrees of freedom whose indicator exceeds one-half of the maximum value or, in the case of the h-convergence program, subdividing elements whose indicator exceeds this value. It has been found that the path followed by the adaptive process is not greatly affected by changing the one-half factor. However, in practice this choice may affect the expense of the solution process and alternative strategies are discussed both by Babuška [7] and Peano [38].

#### 4.1 Examples

4.1.1. *Cantilever with applied end moment.* The error estimators  $\|e\|_{E(\Omega_1)}$  are identical on all elements in the first two meshes

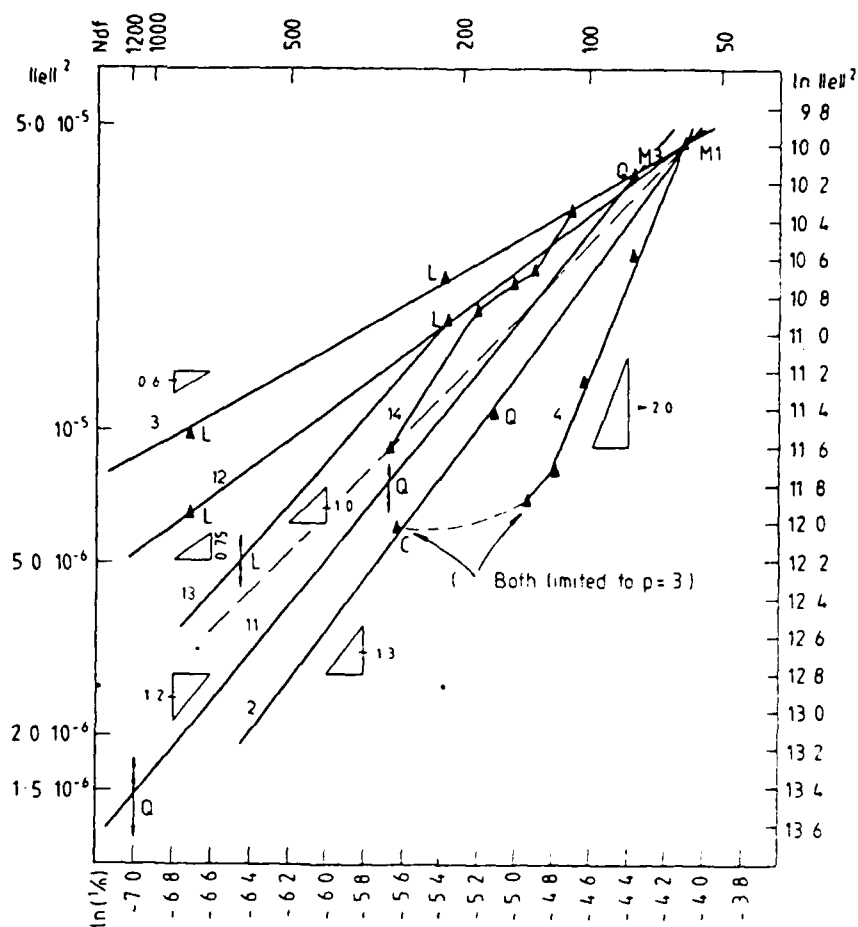
shown in Figure 8, so the sequence of results in Table 1 corresponds to a h-adaptive process. The accuracy of the error estimators indicates that an effective stopping criterion on the basis of the energy norm, stress or displacement, could be defined.

4.1.2. *Small hole in large region.* Both the h and p adaptive processes have been applied to the problem defined in Figure 9 and the results plotted on Figure 11. Notice that the convergence is expressed in terms of number of degrees of freedom and not as usual in terms of h. In the p-version h remains constant and a comparison would not be possible. For 2-D elements  $O(h^2) \approx O(1/N)$  so we expect, for example, for linear elements, slopes  $\leq 1$ .

The plots in Figure 11 can be divided in two groups: convergence using linear trial spaces (adaptive or non-adaptive), and convergence using higher order trial spaces. The first group includes solution extensions nos. 3, 12, 13, 14, and the second group extensions nos. 2, 4, 11, from meshes M1 and M3. Extensions 4, 13, 14 are adaptive and the type of meshes obtained are represented in Figures 12 and 13.

As expected, the rates of convergence are higher for the second group because of the better convergence characteristics of higher order finite elements. Within each group the adaptive solutions are better because there is an intelligent criterion to select the new degrees of freedom.

The adaptive p-extension (extension no.4) tends in the limit to the cubic solution of extension no.2, due to the fact that we set a limit of complete cubic modes in the adaptive p-convergence program. This last result shows that a very accurate solution on p-convergence extensions requires the use of higher order polynomials. The disadvantage is that this leads to a loss in point-wise convergence because of the 'noise' associated with the polynomial oscillations.



#### Extensions

- 2 p - Convergence based on M1, complete refinement
  - 3 h - Convergence based on M1, complete refinement, no geometry updating
  - 4 p - Convergence based on M1, adaptive
  - 11 h - Convergence based on M3, complete refinement
  - 12 h - Convergence based on M1, complete refinement
  - 13 h - Convergence based on M1, adaptive based on local Richardson's extrapolation for error indication
  - 14 h - Convergence based on M1, adaptive based on refinement of top 50% of error indicators in the mesh
- where M1 - mesh shown in Figure 9  
 113 - a mesh of six 9-node elements based on M1

FIG.11. Experimental rates of convergence for small hole in large region

A mixed  $h$  and  $p$  convergence model could therefore be the best strategy. In [14] it is proved that the combined versions produce higher order rates of convergence than either the  $h$  or  $p$ -version by itself, indicating that research in this direction is necessary. We note, however, that the resulting program will have a very complex structure. In this context we reference the work done on the  $h$ -version [43,57] which by itself presents a highly complex situation.

The  $p$ -convergence programs, on the other hand, have a simpler structure but nevertheless more elaborate than the usual finite element codes, since we have to allow for error subroutines, automatic node generation subroutines, and multi-level finite element types.

The meshes obtained for the  $h$  and  $p$  adaptive processes of Extensions 14 and 4 are given in Figure 12 and 13 respectively. Similar refinement near the hole is seen in both cases. Finally, we note from Table 3 that the error estimators for the  $h$  adaptive process given in Figure 12 are again accurate enough to provide a stopping criterion. However, in Section 4.1.2 it was seen that the global energy norm gave little indication of the accuracy of stresses near the hole surface. The local stress error estimator discussed in that section may provide a more practical accuracy test.

Obviously very powerful solution algorithms can be based on these processes. The efficiency of the hierarchic indicators is best shown by returning to the cantilever beam of Figure 8 and considering a tip shear load. The interpolants required on the interfaces between elements such as  $AB$ , in the figure, depends on the Poisson's ratio. With  $\nu = 0.0$  only cubic interpolants in the  $x$ -direction are required; with  $\nu = 0.3$  the exact solution only follows if quadratic as well as cubic interpolants are added on the interfaces. The adaptive process based on the hierarchic error indicators is sensitive to exactly these requirements and the quadratics are left out of the adaptive process for the first problem.

TABLE 3

*Linear Elements (Adaptive Solution - Extension 14)*

|        | ndf | $\ u_h\ _E^2$ | ErrorEstimate<br>$\ e\ _E$ | $\theta$ | ErrorEstimate<br>$\ e\ _E^*$ | $\theta^*$ |
|--------|-----|---------------|----------------------------|----------|------------------------------|------------|
| Step 1 | 60  | 1.333602      | $.583 \times 10^{-2}$      | .85      | $.70 \times 10^{-2}$         | 1.02       |
| Step 2 | 111 | 1.333616      | $.428 \times 10^{-2}$      | .76      | $.52 \times 10^{-2}$         | .92        |
| Step 3 | 133 | 1.333625      | $.382 \times 10^{-2}$      | .80      | $.45 \times 10^{-2}$         | .94        |
| Step 4 | 150 | 1.333626      | $.355 \times 10^{-2}$      | .76      | $.42 \times 10^{-2}$         | .90        |
| Step 5 | 182 | 1.333629      | $.323 \times 10^{-2}$      | .74      | $.38 \times 10^{-2}$         | .87        |
| Step 6 | 287 | 1.333639      | $.295 \times 10^{-2}$      | .98      | $.36 \times 10^{-2}$         | 1.20       |

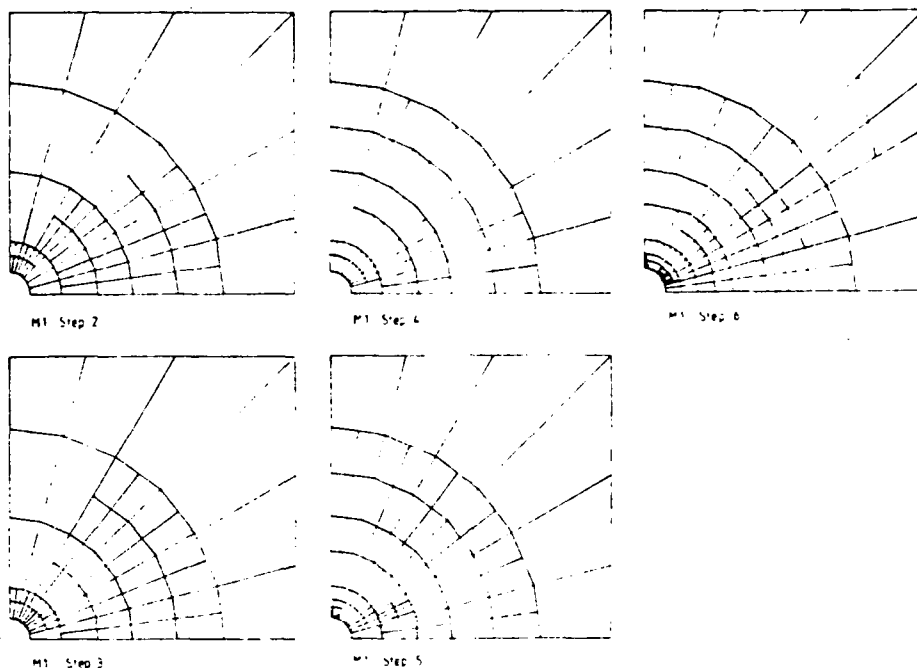


FIG. 12. H-convergence on M1 linear elements - extension 14 (figures not to scale)

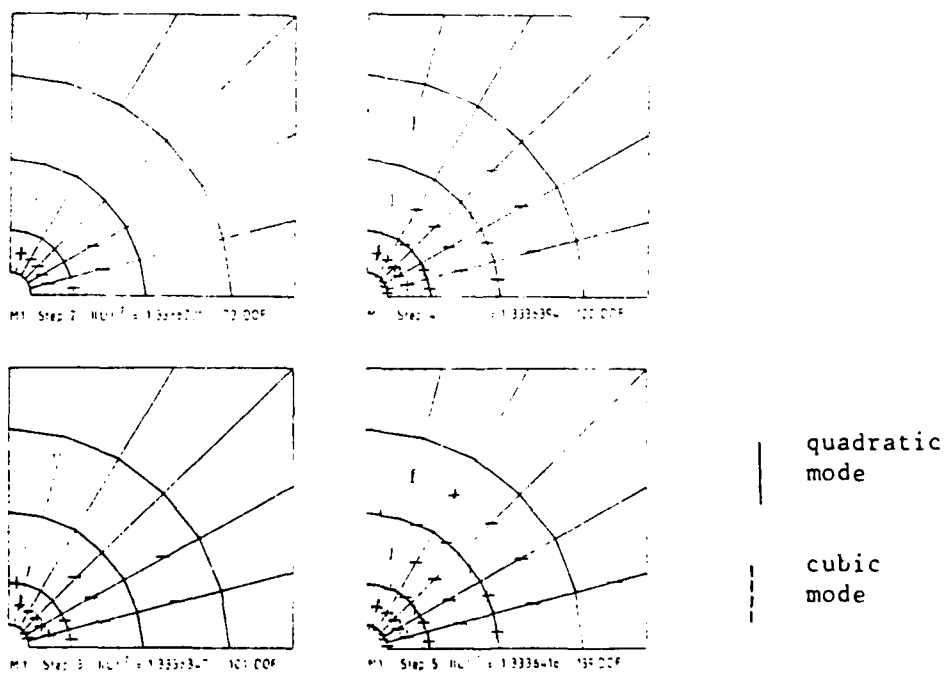


FIG. 13. P-convergence on M1, hierarchical elements - extension 4 (figures not to scale)

## 5. CONCLUDING REMARKS

A general description of 'hierarchical' finite elements, including both  $h$  and  $p$  versions of the finite element method has been presented. There are three main advantages of this approach. Firstly, it leads to improved conditioning of the stiffness matrix and the topology of the stiffness matrix indicates an efficient partitioning for block iteration solution procedures. Secondly, compatibility is easily enforced in meshes with a graded refinement of polynomial order or element size. Finally, in the new generation of adaptive finite element schemes higher order hierarchical modes provide an indicator for the selection of the new degrees of freedom which should be added to the finite element mesh.

We have also demonstrated in this paper that accurate error estimators are available and can be interpreted locally at least for problems with stress concentrations no greater than the circular hole. Projections of the residuals required for the evaluation of these error estimators give the hierarchical indicators which control the adaptive processes. Naturally, the program architecture becomes complex but processes allowing the accuracy required of the solution to be pre-specified are within reach.

## APPENDIX

### PROOF OF THE BOUND OF EQUATION (3.30)

The energy norm of the error is given by

$$||e||_E^2 = - \int_0^L r e \, dx \quad (A.1)$$

$$\text{Take } r = \sum r_i \sin \frac{i\pi x}{L}$$

$$\text{From (3.24) } \frac{d^2 e}{dx^2} = r$$

so that

$$e = - \sum \left( \frac{L}{i\pi} \right)^2 r_i \sin \frac{i\pi x}{L}$$

Substituting in (A.1)

$$||e||_E^2 = + \int_0^L \left[ \sum r_i \sin \frac{i\pi x}{L} \right] \left[ \sum \left( \frac{L}{i\pi} \right)^2 r_i \sin \frac{i\pi x}{L} \right] dx$$



but since

$$\int_0^L \left( \sin \frac{i\pi x}{L} \sin \frac{j\pi x}{L} \right) dx = 0, \quad i \neq j$$

$$\|e\|_E^2 = \frac{L^2}{\pi^2} \int_0^L \left[ \sum \frac{r_i}{i} \sin \frac{i\pi x}{L} \right]^2 dx$$

$$\leq \frac{L^2}{\pi^2} \int_0^L \left[ \sum r_i \sin \frac{i\pi x}{L} \right]^2 dx$$

$$= \frac{L^2}{\pi^2} \int_0^L r^2 dx$$

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